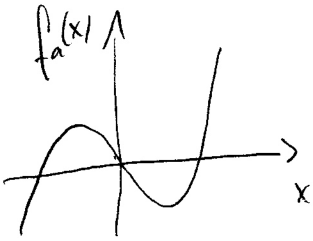
1. (a): $f_a(x) = x^3 + ax$

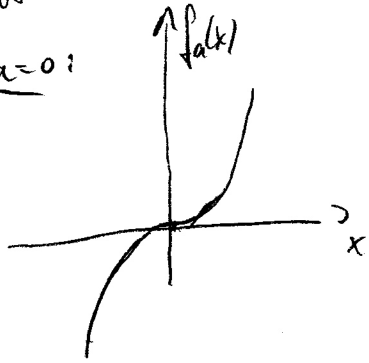
$= x(x^2 + a)$

$= 0$ for $x = 0$ or $x = \pm\sqrt{-a}$

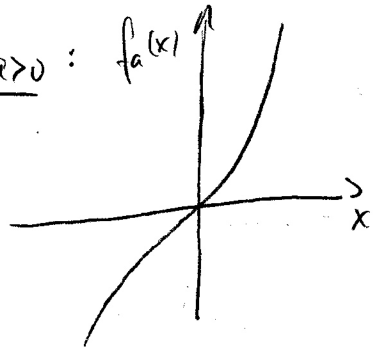
a<0:



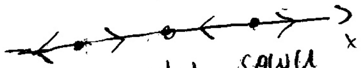
a=0:



a>0:

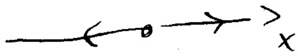


phase portraits



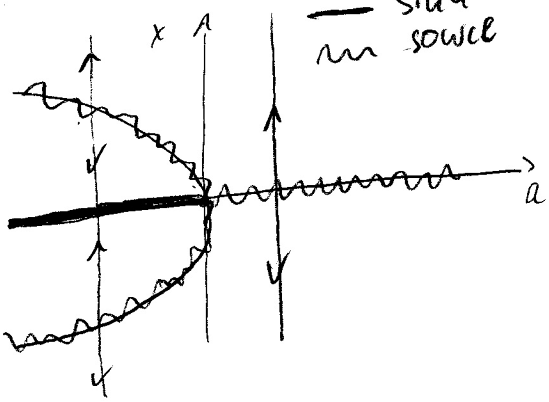
source sink source

source



source

bifurcation diagram:



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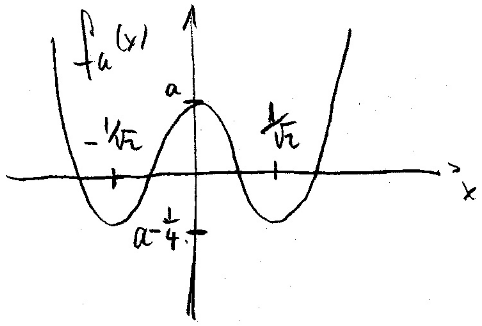
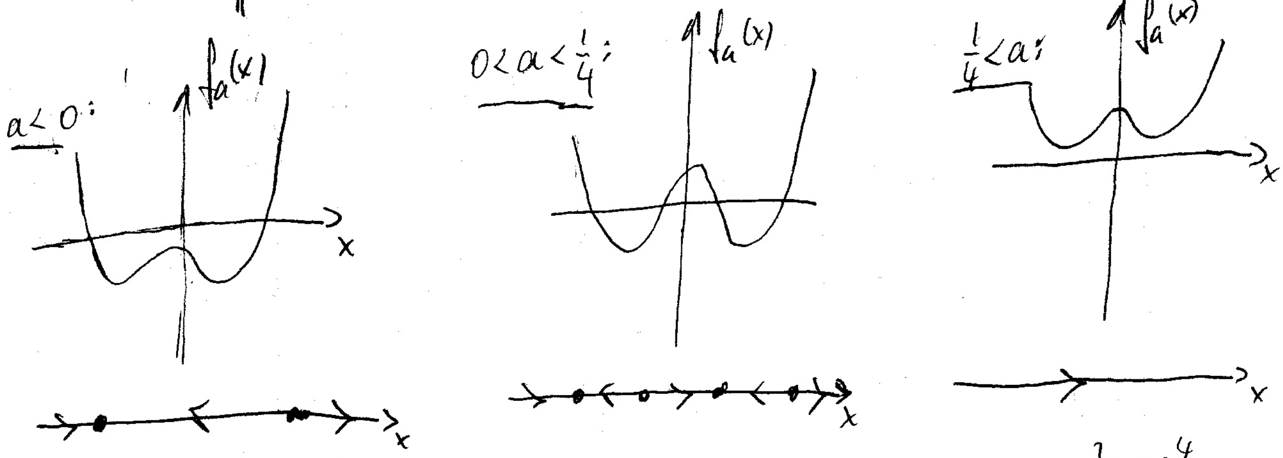
subcritical pitchfork bifurcation

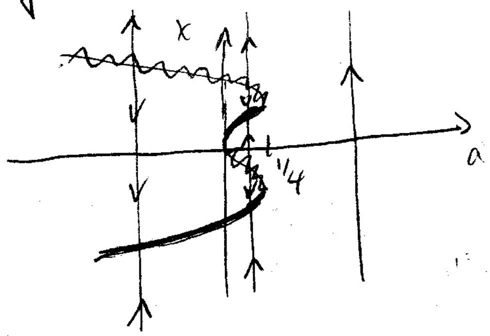
for $f_a(x) = x^3 - ax$, replace a by $-a$

(b) $f_a(x) = x^4 - x^2 + a$





bifurcation diagram is given by $x^4 - x^2 - a = 0 \iff a = x^2 - x^4$



: source

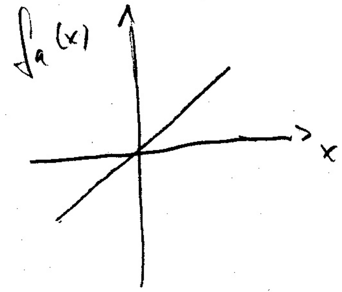
: sink

∴ saddle-node bifurcations at $a=0$ and $a=\frac{1}{4}$

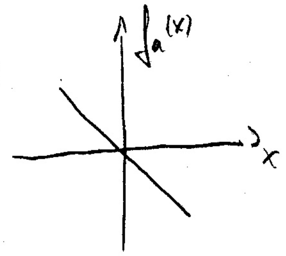
for $f_a(x) = x^4 - x^2 - a$, replace a by $-a$

(c) $f_a(x) = \sin x + ax = 0 \iff a = -\frac{\sin x}{x}$ or $x = 0$

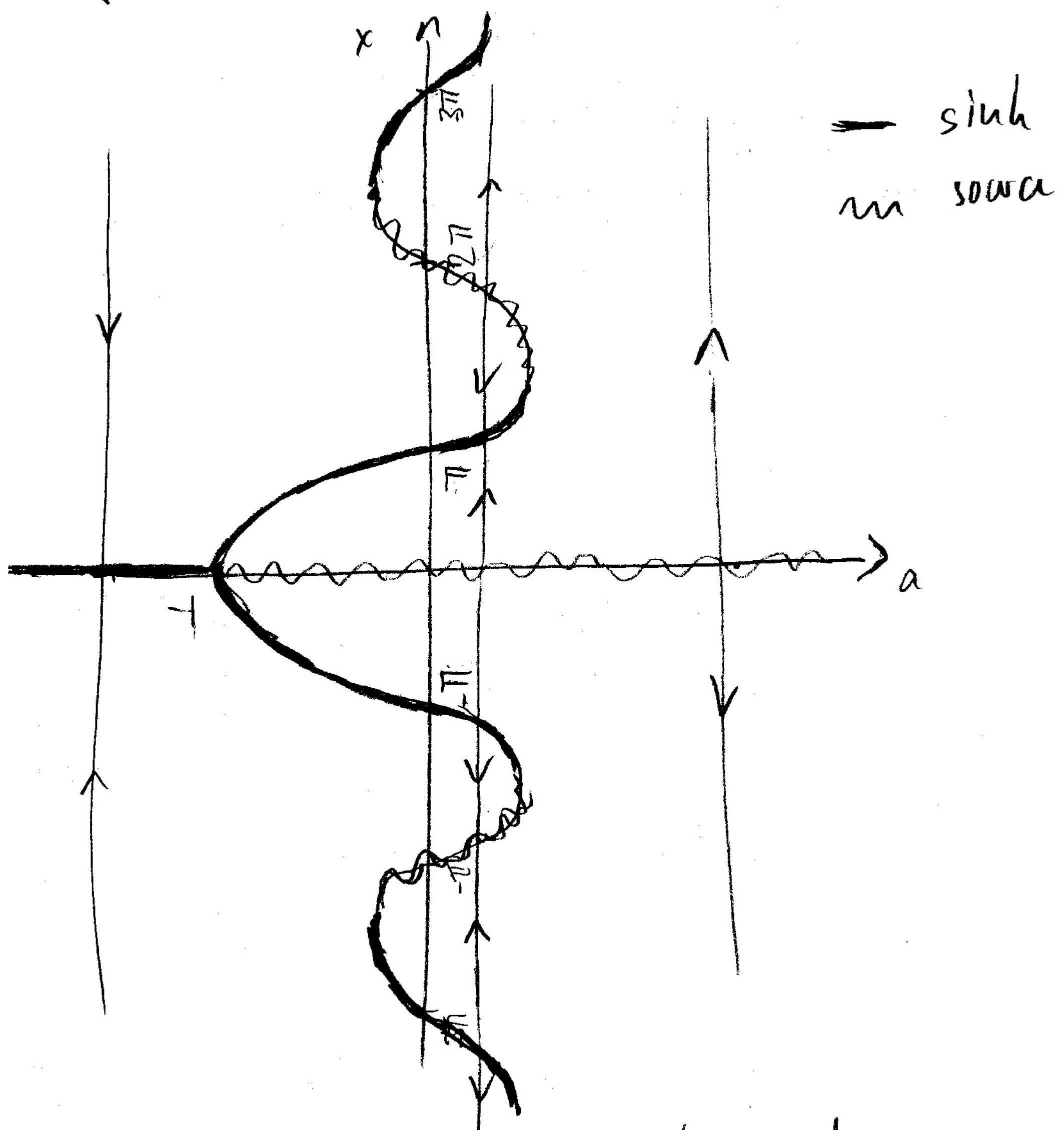
for $a \gg 1$:



for $a \ll -1$:



bifurcation diagram:



supercritical pitchfork bifurcation at $a=-1$,
all other bifurcations are saddle-node bifurcations
for $f_a(x) = \sin x - ax$, replace a by $-a$

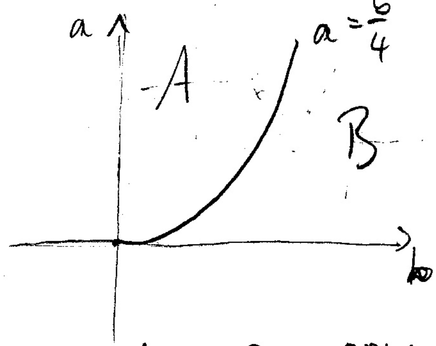
2. (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$ with $M = \begin{pmatrix} 0 & a \\ -1 & -b \end{pmatrix}$

M has eigenvalues $\lambda_\pm = -\frac{b}{2} \pm \frac{\sqrt{b^2-4a}}{2}$

$Re(\lambda_+), Re(\lambda_-) < 0$ as $b > 0$

$\Rightarrow$ equilibrium at $(0,0)$ is asymptotically stable

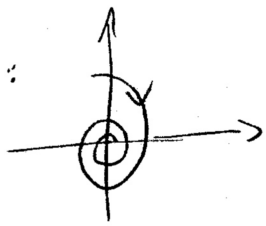
(b) λ_+, λ_- real for $b^2 - 4a > 0$. Otherwise λ_+, λ_- compl. (not real)



in region A: $\lambda_\pm$ complex (not real)

canon. form: $\begin{pmatrix} Re\lambda_+ & Im\lambda_+ \\ -Im\lambda_+ & Re\lambda_+ \end{pmatrix} = \begin{pmatrix} -\frac{b}{2} & \frac{1}{2}\sqrt{4a-b^2} \\ -\frac{1}{2}\sqrt{4a-b^2} & -\frac{b}{2} \end{pmatrix}$

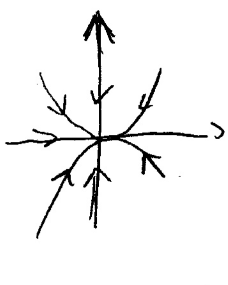
phase portrait:



in region B: $\lambda_\pm$ real

canon. form: $\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} -\frac{b}{2} + \frac{\sqrt{b^2-4a}}{2} & 0 \\ 0 & -\frac{b}{2} - \frac{\sqrt{b^2-4a}}{2} \end{pmatrix}$

phase portrait:



(c) system represents harmonic oscillator with damping

For $b=0$, system is Hamiltonian.

Namely for $H(x,y) = \frac{1}{2} a y^2 + \frac{1}{2} x^2$

$$x' = \frac{\partial H}{\partial y}$$

$$y' = -\frac{\partial H}{\partial x}$$

$H(x,y)$ is constant of motion

For $b>0$, choose Lyapunov function

$$L(x,y) = H(x,y) = \frac{1}{2} a y^2 + \frac{1}{2} x^2$$

$$\Rightarrow \quad \overset{\circ}{L}(x,y) = L_x x' + L_y y' = -aby^2$$

L is Lyapunov function on $\mathbb{R}^2$.

For $d>0$, set $D_d = \{ L \le d \}$

$$= \{ (x,y) \in \mathbb{R}^2 \mid \tfrac{1}{2} x^2 + \tfrac{1}{2} a y^2 \le d \}$$

area enclosed by ellipse $\frac{1}{2}x^2 + \frac{1}{2} a y^2 = d$

D_d is compact and positively invariant (as D_d is enclosed by level set of L and L is a Lyapunov function).

Suppose $(x(t), y(t))$ solution in D_d with $L(x(t), y(t)) = \text{const.}$

$\Rightarrow \quad \overset{\circ}{L}(x(t), y(t)) = -ab\,y^2(t) \equiv 0 \Rightarrow y(t) \equiv 0$ f.a. t

$\Rightarrow \quad 0 \equiv y'(t) = -x(t) - b\,y(t) = -x(t)$

$\Rightarrow \quad (x(t), y(t)) \equiv (0,0)$ which is the equilibrium solution. Hence there is no other solution in D_d on which L is constant.

$\Rightarrow$ By Lasalle's Invariance Principle is the equilibrium at $(0,0)$ asymptotically stable and D_d belongs to the basin of attraction. As this holds for any $d>0$, we conclude that the basin of attraction is $\mathbb{R}^2$.

3. $\bar{X}' = F(x)$ gradient system

$\Rightarrow \exists f : \mathbb{R}^k \to \mathbb{R}$, s.t. $F(x) = \nabla f(x)$

Let $\bar{X}(t)$ be a recurrent solution.

$\Rightarrow \exists$ sequence of times (t_n) with $t_n \to \infty$ (monotonically) as $n \to \infty$.

Note $\frac{d}{dt} f(\bar{X}(t)) = \nabla f(\bar{X}(t)) \cdot \nabla f(\bar{X}(t))$

$= \|\nabla f(\bar{X}(t))\|^2$

which is strictly positive if $\nabla f(\bar{X}(t)) \neq 0$ and zero only if $\nabla f(\bar{X}(t)) = 0$, i.e. if $\bar{X}(t)$ is equilibrium.

Suppose $\bar{X}(t)$ no equilibrium.

Then $f(\bar{X}(t_0)) < f(\bar{X}(t_n)) \to f(\bar{X}(t_0))$ as $n \to \infty$.

which is a contradiction.

So $\bar{X}(t)$ is equilibrium solution.

4. Set $g(x,\lambda) := f_\lambda(x)$

$\Rightarrow g(x_0,\lambda_0) = 0$ and $\frac{\partial g}{\partial x}(x_0,\lambda_0) = f'_{\lambda_0}(x_0) - 1 \neq 0$

By Implicit Function Th^m, exist intervals an interval $I \subset \mathbb{R}$ forming a neighb. of x_0, an interval $J \subset \mathbb{R}$ forming a neighb. of λ_0 and a function $F : J \to I$, $\lambda \mapsto F(\lambda)$

s.t. $g(x,\lambda) = 0$ for $(x,\lambda) \in I \times J$

$\Leftrightarrow x = F(\lambda)$, $\lambda \in J$

let $|f'_{\lambda_0}(x_0)| = \nu > 1$ and $1 < K < \nu$.

By continuity of the derivatives of f w.r.t. x and λ we can make the neighb. I of x_0 and the neighb. J of λ_0 sufficiently small to have

$$|f'_\lambda(x)| > K \quad \text{f.a. } (x,\lambda) \in I \times J$$

let $(x,\lambda) \in I \times J$ with $x \neq F(\lambda)$.

Then: $$\frac{f_\lambda(x) - F(\lambda)}{x - F(\lambda)} = \frac{f_\lambda(x) - f_\lambda(F(\lambda))}{x - F(\lambda)}$$

$$= f'_\lambda(c) \quad \text{with } c \in I$$

by Mean Value Th^m.

$$\Rightarrow |f_\lambda(x) - F(\lambda)| > K\,|x - F(\lambda)|$$

Similarly

$$|f_\lambda^2(x) - 7(\lambda)| > K|f_\lambda(x) - 7(\lambda)| > K^2|x - 7(\lambda)|$$

and by induction

$$|f_\lambda^n(x) - 7(\lambda)| > K^n|x - 7(\lambda)| \quad \text{f.a. } n \in \mathbb{Z}_{>0}$$

As $K^n|x - 7(\lambda)| \to \infty$ as $n \to \infty$

it must hold that

$f_\lambda^n(x) \notin I$ for some $n > 0$.

5. Set $g(x,a) = f(x,a) - x$

$$\Rightarrow \quad g(x_0, a_0) = f(x_0, a_0) - x_0 = 0$$

$$\text{and} \quad \frac{\partial g}{\partial a}(x_0, a_0) = \frac{\partial f}{\partial a}(x_0, a_0) \neq 0$$

By Implicit Function Th^m, there exists I neighb. of x_0, J neighb. of a_0 and

$\lambda : I \to J$ s.t. $g(x,a) = f(x,a) - x = 0$ for $(x,a) \in I \times J$

$x \mapsto \lambda(x)$ $\iff a = \lambda(x)$

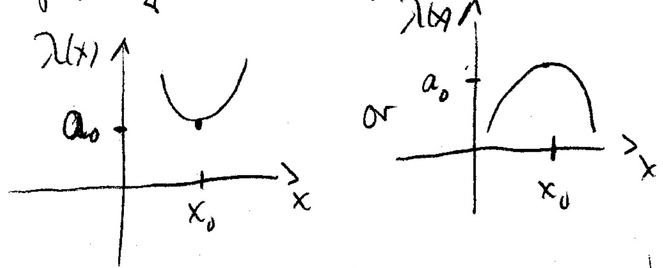
$$\lambda'(x) = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial a}} = -\frac{\frac{\partial f}{\partial x} - 1}{\frac{\partial f}{\partial a}} = 0 \quad \text{for } (x,a) = (x_0, a_0)$$

$$\lambda''(x) = -\frac{d}{dx}\,\frac{\frac{\partial f}{\partial x} - 1}{\frac{\partial f}{\partial a}} = -\frac{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial a \partial x}\lambda'\right)\frac{\partial f}{\partial a} - \left(\frac{\partial f}{\partial x} - 1\right)\left(\frac{\partial^2 f}{\partial x \partial a} + \frac{\partial^2 f}{\partial a^2}\lambda'\right)}{\left(\frac{\partial f}{\partial a}\right)^2}$$

$$= -\frac{\frac{\partial^2 f}{\partial x^2}}{\frac{\partial f}{\partial a}} \quad \text{for } (x,a) = (x_0, a_0)$$

$$\neq 0$$

so the graph of $\lambda(x)$ locally looks like



These are the bifurcation diagrams of saddle-node bifurcations

6. $h \circ f = g \circ h$ means that the following diagram commutes:

$$\begin{tikzcd} [0,1] \arrow[r, "f"] \arrow[d, "h"'] & [0,1] \arrow[d, "h"] \\ [0,1] \arrow[r, "g"'] & [0,1] \end{tikzcd}$$

Let $x_0, x_1, x_2, \dots$ be an orbit of f, i.e. $x_{k+1} = f(x_k)$
$k \in \mathbb{Z}_{\geq 0}$

Set $y_k := h(x_k)$, $k \in \mathbb{Z}_{\geq 0}$

Then: $g(y_k) = g \circ h(x_k) = h \circ f(x_k) = h(x_{k+1}) = y_{k+1}$

f.a. $k \in \mathbb{Z}_{\geq 0}$.

Hence: $y_0, y_1, y_2, \dots$ is an orbit of g.

1. Show that periodic points of g are dense:

Let $\Pi_f := \{x \in [0,1] \mid x \text{ period. point of } f\}$

$\Rightarrow \Pi_f$ dense in $[0,1]$.

Note that by the arguments above that if x is a periodic point of f then $y := h(x)$ is a periodic point of g.

Show: $h(\Pi_f) \subset [0,1]$ dense:

Let $U \in [0,1]$ open. To be shown: $h(\Pi_f) \cap U \neq \emptyset$.

h continuous $\Rightarrow$ $h^{-1}(U) =: \hat{U}$ open

$\Rightarrow$ $\hat{U} \cap \Pi_f \neq \emptyset$ as Π_f is dense in $[0,1]$.

$\Rightarrow$ $\exists x \in \hat{U} \cap \Pi_f$.

$\Rightarrow$ $y := h(x) \in U$ and $y \in f(\Pi_f)$

i.e. $y \in h(\Pi_f) \cap U$.

So: $h(\Pi_f) \cap U \neq \emptyset$

2. Show that g is transitive:

Let $U, \bar{V} \subset [0,1]$ open. To be shown:

$\exists n \in \mathbb{Z}_{>0}$ s.t. $g^n(U) \cap \bar{V} \neq \emptyset$.

Let $\hat{U} := h^{-1}(U)$ and $\hat{\bar{V}} := h^{-1}(\bar{V})$.

Then $\hat{U}$ and $\hat{\bar{V}}$ are open as h is continuous.

As f is transitive $\exists n \in \mathbb{Z}_{>0}$ s.t.

$f^n(\hat{U}) \cap \hat{\bar{V}} \neq \emptyset$, i.e. $\exists x \in \hat{U}$ s.t. $f^n(x) \in \hat{\bar{V}}$.

Set $y = h(x)$.

$\Rightarrow y \in U$ and $g^n(y) = g^n(h(x)) = h(f^n(x)) \in h(\hat{\bar{V}}) = \bar{V}$.

Hence: $g^n(U) \cap \bar{V} \neq \emptyset$.

3. Show that g has sensitive dependence on initial conditions:

For a map on interval like g this follows from points 1. and 2.